STRAIGHTENING-OUT AND SEMIRIGIDITY IN ASSOCIATIVE ALGEBRAS

BY

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Introduction. Let $A = N \oplus S$ be a finite-dimensional associative algebra over the perfect field k, here written as the semidirect sum of its radical N and a separable Wedderburn factor S. We do not require that A have a multiplicative unit. Denote the multiplication in A by $\pi(x, y) = xy$. Suppose we are given a deformation of the algebra structure on A via the associative bilinear map $\pi_t(x, y) = xy + tF_1(x, y) + t^2F_2(x, y) + \cdots$; here π_t is the multiplication in the generic element A_t of a one-parameter family of deformations of A in the sense of Gerstenhaber [3, p. 62]. (Recall that A_t is an algebra over the scalar field $K = k((t))$, the field of power series over k in an indeterminate t.) It is natural to ask how the structure of the deformed algebra A_t compares with that of A (or, to be technically precise, with that of A_K, the algebra with multiplication π but scalars extended to K). For example, if $A = S$ is separable semisimple, then A is rigid and A_t is isomorphic to A_K. At the opposite extreme, however, if $A = N$ is nilpotent, then it is highly deformable (see a forthcoming paper), and A_t may be quite unlike A_K.

In the first part of this paper we state a result (Theorem 1) which straightens-out (as far as possible) the deformation π_t. That is, we construct a deformation μ_t equivalent to π_t (see [3, p. 65]) but better behaved as follows: μ_t does not subject the underlying subspaces of the radical N and Wedderburn factor S to a meaningless rotation, and μ_t respects the rigidity of the semisimple S by not introducing trivial deformations here. This enables us to compare A_t (multiplication μ_t now) with A_K. The view thus afforded leads us to define semirigidity as a natural generalization of rigidity; see §2. Briefly, semirigid algebras are those which cannot become more semisimple through deformation. In §3 and §4 we examine in some detail the two types of deformation which increase semisimplicity. Our analysis yields explicit deformation formulae and a method of constructing semirigid algebras.

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1. **The straightening-out theorem.** Roughly put, this will tell us (i) the radical N_K of A_K shrinks under deformation to that of A_t, while a Wedderburn factor

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of A_t contains S_K and absorbs part of N_K, (ii) the multiplication in the subalgebra S_K is unchanged, (iii) the two-sided regular action of S_K on N_K is unchanged.

In what follows we work in the context of the Wedderburn Principal Theorem; that is, all algebras have a semidirect sum decomposition (radical) $\oplus$ (semisimple); this is always the case in characteristic zero. Also, V will denote the underlying k-space of A; we write $A = \text{alg}(V, \pi)$ and similarly for other algebras. For any k-space U we denote the scalar extension $U \otimes_k K$ by U_K. It will often be convenient to suppose U is imbedded in U_K. All other definitions and notations are in [3]. The above description can now be made precise.

THEOREM 1. *Let $A = N \oplus S$ and π, π_t be as above. Then*

(a) *there exists a K-space decomposition $N_K = M_K \oplus T$, where M (resp. T) is a k- (resp. K-) subspace of N (resp. N_K), and*

(b) *there exists a generic deformation $\mu_t(x, y) = xy + tG_1(x, y) + \cdots$ of A equivalent to π_t such that*

(c) *$M = \text{alg}(M, \pi)$ is a nilpotent ideal of A,*

(d) *$M_t = \text{alg}(M_K, \mu_t)$ is the radical of the K-algebra $A_t = \text{alg}(V_K, \mu_t)$,*

(e) *$\mu_t(x, y) = xy$ for all $x, y \in S_K$,*

(f) *$\mu_t(x, z) = xz$ and $\mu_t(z, x) = zx$ for $x \in S_K$, $z \in N_K$,*

(g) *T is an S_K-bimodule in both A_K and A_t,*

(h) *$W_t = \text{alg}(T \oplus S_K, \mu_t)$ is a semisimple Wedderburn factor of A_t, so that $A_t = M_t \oplus W_t$.*

Proof. This is lengthy but elementary; we omit many details. It is important to construct intermediate equivalences in the proper order. (i) First we deal with the radicals, proving (c), (d) and part of (a). Let $\zeta_1, \ldots, \zeta_r$ be a K-basis for rad (π_t), a subspace of V_K. Without loss $\zeta_i = z_i +$ (powers of t with coefficients in V) and the z_i are k-independent. Using the definition of [7, p. 140], one shows that the z_i are in the radical N of A. Define M to be the k-space with basis $z_1, \ldots, z_r$. M is an ideal for the multiplication π because rad (π_t) is for π_t; this gives (c) above. Now extend $z_1, \ldots, z_r$ to a k-basis of V which is thereby a K-basis of V_K. Define $\Phi_t(\zeta_i) = z_i$ and extend to all of V_K using the k-basis so that Φ_t has the form $\text{id} + t\phi_1 + \cdots$. The multiplication $\Phi_t \circ \pi_t \circ (\Phi_t^{-1} \times \Phi_t^{-1})$ is equivalent to π_t and satisfies (d) above.

(ii) Now suppose π_t has the properties established in (i). We obtain (e) and (f) by applying a miniature deformation theory of algebra homomorphisms which we now sketch. Let A, B be k-algebras, B with unit 1, and let $f: A \to B$ be an algebra homomorphism which need not map unit to unit. Thus B is an (A, f)-bimodule. A *deformation* of f is given by a K-algebra homomorphism $f_t: A_K \to B_K$ of the form $f_t = f + tF_1 + \cdots$, where $F_i: A \to B$ is k-linear. We say f_t is *trivial* iff there is an inner automorphism $\beta_t: B_K \to B_K$ given by conjugating with an element $1 + tb_1 + \cdots$ of B_K such that $f_t = \beta_t \circ f$. We say f is *rigid* iff all such deformations f_t are trivial. Now let $H^*(A, f, B)$ denote Hochschild cohomology of A with coefficients in the (A, f)-bimodule B. One proves, in analogy with [3, p. 65], the

LEMMA (NIJENHUIS). *If* $H^1(A, f, B)=(0)$, *then* f *is rigid.*

To obtain (e) and (f), form $\Sigma = S \otimes_k S^{\mathrm{op}}$ and map $f\colon \Sigma \to \mathrm{End}\,(V)$ by $(f(x \otimes x'))y = xyx'$ for all $x, x' \in S$ and $y \in A$. Now write $\pi_t(y, y') = y * y'$ for all $y, y' \in V_K$. Then $f_t\colon \Sigma_K \to \mathrm{End}\,(V)_K$ defined by $(f_t(x \otimes x'))y = x * y * x'$ is a deformation of f. Since Σ is separable semisimple, f_t must be trivial. Thus there is a K-linear automorphism Ξ_t of V_K of the form $\Xi_t(x) = x + t\xi_1(x) + \cdots$ such that the composition $\Xi_t^{-1} \circ \pi_t \circ (\Xi_t \times \Xi_t)$ satisfies (e) and (f) as well as (d). We call this multiplication μ_t.

(iii) Now we obtain W_t, T, (a), (g) and (h). Let W_t be a Wedderburn factor containing the subalgebra S_K of $A_t = \mathrm{alg}\,(V_K, \mu_t)$. Define the K-subspace T as $N_K \cap W_t$. By dimension, (a) holds. Also, by (f), $\mu_t(S_K, T) = \pi(S_K, T) \subset N_K$, and since S_K and T are contained in W_t we have $\mu_t(S_K, T) \subset W_t$. Thus $\mu_t(S_K, T) \subset T$ and (g) follows. The definition of T yields (h), and the theorem is proved.

A corollary (not new) of (e) is: *The dimension of the radical does not increase under deformation.*

The following illustrates the theorem, underlining the facts that the radical decomposes over K but not, in general, over k and also that T need not be a subalgebra of A_K. Let $A = N$ have basis x, y, z over k and multiplication $xy = z$, all other products of basis elements zero. Now obtain A_t over K by setting $\pi_t(x, y) = xy = z$, $\pi_t(x, x) = tx$, $\pi_t(y, y) = ty$, $\pi_t(x, z) = \pi_t(z, y) = tz$, other products zero. Let E denote the (rigid) algebra of two-by-two matrices over K with lower left-hand corners zero and e_{11}, e_{12}, e_{22} the usual basis. Then the map $t^{-1}x \to e_{11}$, $t^{-1}y \to e_{22}$, $t^{-2}z \to e_{12}$ is a K-algebra isomorphism $A_t \to E$. In the language of the theorem, $A_t = M_t \oplus T$, where z is a k-basis for M and $t^{-1}x - t^{-2}z$, $t^{-1}y$ is a K-basis of orthogonal idempotents for $T = W_t$. And T clearly has no k-basis. For the nilpotent deformations of A, see [3, p. 91].

An analogous theorem for Lie algebras has been obtained by Page and Richardson [6], using the geometric methods of [5]. The proof of parts (e) and (f) of Theorem 1 above shows, in fact, that if S is any subalgebra of A, not necessarily semisimple, such that the group $H^1(S \otimes_k S^{\mathrm{op}}, \mathrm{End}\,(V))$ is zero, then S is "stable" under deformations of A, as is the regular action of S on the bimodule A.

We shall use Theorem 1 repeatedly in §3 and §4 to examine the types of deformation introduced there. However, let us exercise it immediately (i) to construct some rigid algebras, and (ii) to prove the algebra of all upper-triangular matrices rigid.

(i) An example reveals the method of construction. Let S_1, S_2 be any separable k-algebras with units e_1, e_2. Let N be any (S_1, S_2)-bimodule. Form the associative algebra $A = N \oplus S$, with S the ideal direct sum $S_1 \oplus S_2$, by defining $N^2 = Ne_1 = e_2N = (0)$. Then A is rigid: for let $\mu_t(x, y) = x * y$ determine a straightened-out deformation of A. By Theorem 1 only products of elements $z, z' \in N$ may deform. But $z * z' = (ze_2) * (e_1z') = z(e_2e_1)z' = 0$, whence μ_t is trivial as desired. The idea here, of course, is that the multiplication in N is constrained by the rigid action of S

to be zero both before and after deformation. This observation may be stated as a theorem. We have $A=N\oplus S$ with $S=S_1\oplus\cdots\oplus S_s$ the direct sum of separable simple S_α with unit e_α. Let $N_{\alpha\beta}=e_\alpha N e_\beta$. Form $e=e_1+\cdots+e_s$ which is idempotent but need not be a two-sided identity. Now define

$$N_{0\alpha} = \{z\in N \mid ez = 0 \text{ and } ze_\alpha = z\}$$

and likewise for $N_{\alpha 0}$ and N_{00}. Arguing as above we have

THEOREM 1.1. *Let $A=N\oplus S$ over the perfect field k. Then $N^2=(0)$ and A is rigid iff*

(a) $N_{\alpha\alpha}=(0)$,

(b) *if* $N_{\alpha\beta}>(0)$, *then* $N_{\beta\alpha}=(0)$, *and*

(c) *if* $N_{\alpha\beta}>(0)$ *and* $N_{\beta\gamma}>(0)$, *then* $N_{\alpha\gamma}=(0)$ *for* $\alpha, \beta, \gamma=0, 1, \ldots, s$.

(ii) Now we consider an algebra with $N^2>(0)$. Let $\nabla=\nabla(r,k)$ be the algebra of all r by r matrices over k which have only zero entries below the main diagonal. ∇ has the usual k-basis $e_{\alpha\beta}$, with $1\leqq\alpha\leqq\beta\leqq r$, satisfying $e_{\alpha\beta}e_{\beta\gamma}=e_{\alpha\gamma}$ while $e_{\alpha\beta}e_{\gamma\delta}=0$ for $\beta\neq\gamma$.

The following seems to be widely believed. We offer a constructive proof.

THEOREM 1.2. *$\nabla(r,k)$ is rigid.*

Proof. Let a deformation μ_t be straightened-out as in Theorem 1. Then products involving the $e_{\alpha\alpha}$ are unchanged, while by part (f) we conclude that $e_{\alpha\beta} * e_{\beta\gamma} =\mu_t(e_{\alpha\beta}, e_{\beta\gamma})=\xi_{\alpha\beta\gamma}e_{\alpha\gamma}$, with $\xi_{\alpha\beta\gamma}\in k[[t]]$ and constant term $\xi_{\alpha\beta\gamma}(0)=1$. To prove μ_t trivial, one notes that it suffices to prove the existence of $\eta_{\alpha\beta}\in k[[t]]$ with constant term $\eta_{\alpha\beta}(0)=1$ such that the "perturbed" basis $e'_{\alpha\beta}=\eta_{\alpha\beta}e_{\alpha\beta}$ satisfies $e'_{\alpha\beta} * e'_{\beta\gamma}=e'_{\alpha\gamma}$. Clearly, this is possible when $r=1, 2, 3$. We suppose the theorem is true in rank $r-1$ and proceed by induction. Thus the subalgebra B of ∇ with basis $e_{\alpha\beta}$, $1\leqq\alpha\leqq\beta\leqq r-1$, is rigid and the restriction of μ_t to B is trivial. Hence there exist $\eta_{\alpha\beta}$ and $e'_{\alpha\beta}$ as required, except when $\beta=r$; now we treat the right-hand column. First put $\eta_{1r}=1$, so that $e'_{1r}=e_{1r}$. Now define η_{2r} so that $e'_{12} * e'_{2r}=e'_{1r}$, where $e'_{2r}=\eta_{2r}e_{2r}$. This is always possible. (Note that only e'_{12} multiplies e_{2r} on the left, except for the left identity $e'_{22}=e_{22}$.)

Now, in the third row, define η_{3r} so that $e'_{23} * e'_{3r}=e'_{2r}$ defined previously. Then we note that $e'_{13} * e'_{3r}=e'_{12} * (e'_{23} * e'_{3r})=e'_{1r}$, the desired product. Continue down the column, always defining η_{pr} so that $e'_{qp} * e'_{pr}=e'_{qr}$ where $q=p-1$. The theorem follows.

The elementary reduction of §4, (ii) immediately yields the rigidity of the "block upper-triangular" matrix algebra obtained from ∇ by replacing each $e_{\alpha\alpha}$ by an algebra S_α of all r_α by r_α matrices over k and each $e_{\alpha\beta}$ with $\alpha<\beta$ by an $r_\alpha r_\beta$-dimensional (S_α, S_β)-bimodule.

(ADDENDUM. Both David Knudson and the referee have pointed out that ∇ has global dimension one, whence $H^2(\nabla,\nabla)=0$ and Theorem 1.2. Cf. Eilenberg, Rosenberg and Zelinsky in Nagoya Math J. **12** (1957), or M. Auslander's Brandeis notes "*Rings, modules and homology.*")

2. **Semirigidity.** We noted above that deformation cannot increase the dimension of the radical. Let us generalize the notion of rigidity as follows: we say that A is *semirigid* over k iff A admits no deformation over k which decreases the dimension of the radical. Thus, in the language of Theorem 1, A is semirigid iff for all deformations μ_t the corresponding subspace T (which measures the growth in semisimplicity) is zero. Semirigid algebras are natural objects of study for the following reasons: (a) every deformation thereof is simply the deformation of a nilpotent multiplication, and (b) every associative algebra can be deformed into a semirigid algebra,—but not necessarily into a rigid algebra. (See a forthcoming paper [2].) As we are about to see, moreover, certain tests for semirigidity involve only rather coarse structural invariants of A (e.g. dimension of certain subspaces, solvability of some finite linear relations) which often can be readily checked for a given multiplication table.

Finally, one may phrase the problem of finding all rigid algebras as follows: select from among the semirigid algebras those algebras $A=N\oplus S$ whose radicals N, constrained by the two-sided action of S (rigid by Theorem 1), admit no deformations. The basic Rigidity Theorem has its analogue here.

To facilitate the discussion, from now on $A=N\oplus S$ will have unit e and the scalar field k will be perfect. Also $S=S_1\oplus\cdots\oplus S_s$ is an ideal direct sum of simple algebras, with each S_α by *fiat* a total r_α by r_α matrix algebra over k with unit e_α; we do not treat division algebras over k. Note $e=e_1+\cdots+e_s$. Writing $N_{\alpha\beta}$ for $e_\alpha N e_\beta$, we have $N=\bigoplus N_{\alpha\beta}$ with $\alpha,\beta=1,\ldots,s$; this is a so-called Peirce decomposition of the radical into a direct sum of subalgebras. Note $N_{\alpha\beta}N_{\gamma\delta}=(0)$ if $\beta\neq\gamma$.

Now we observe, using Theorem 1, that a deformation with $T\neq(0)$ may combine the following two phenomena:

(I) A subalgebra of the radical N_K may deform into a total matrix algebra. In particular, a radical element becomes, in the new multiplication, an idempotent. Example: let A have basis z, e with $z^2=0$, $e^2=e=$the two-sided identity (unit) of A. Deform via $\mu_t(z,z)=tz$, other products among basis elements as before. Then the K-algebra A_t is isomorphic to the ideal direct sum $K\oplus K$, and the element $u=t^{-1}z$ is an idempotent for μ_t.

(II) Under deformation, a subspace of the radical N_K (in fact, of T) may coalesce with two or more matrix subalgebras of S_K to form one larger matrix algebra. In this case, no idempotents are produced from radical elements alone; all idempotents of A_t "involve" idempotents of S_K.

Example: Let A be four-dimensional, with z_{12}, z_{21} a basis for N, satisfying $N^2=(0)$, and e_1, e_2 orthogonal idempotents in S. For distinct $\alpha,\beta=1,2$ let $e_\alpha z_{\alpha\beta}=z_{\alpha\beta}e_\beta=z_{\alpha\beta}$ (this is essential) with other products zero. Now obtain A_t via the deformation $\mu_t(z_{\alpha\beta},z_{\beta\alpha})=t^2e_\alpha$, with other products as before. We see that the map $t^{-1}z_{\alpha\beta}\to e_{\alpha\beta}$, $e_\alpha\to e_{\alpha\alpha}$ gives a K-isomorphism $A_t\to M(2,K)=$the algebra of 2 by 2 matrices over K with usual basis $e_{\alpha\beta}$.

Following this analysis we make two more definitions. Let π_t determine a deformation of A and suppose, further, that π_t is straightened-out as in the conclusion of Theorem 1. Thus $A_t=\text{alg}(V_K, \pi_t)=M_t \oplus T \oplus S_K$. We say that π_t gives a deformation of *Type* I iff some element of T is an idempotent for π_t. We say that π_t gives a deformation of *Type* II iff $T\neq(0)$ but (as in example (II)) no element of T is an idempotent for π_t. Thus A is not semirigid iff it admits a deformation of either type. We shall study semirigidity from this point of view in the next two sections.

3. **Deformations of Type I.** First we present a readily applicable necessary condition for Type I deformation. The argument is clarified by writing the deformed product $\pi_t(x, y)$ as $x * y$. It is essential to note that if π_t is straightened-out, then $N_{\alpha\beta} * N_{\gamma\delta}=(0)$ for $\beta\neq\gamma$.

THEOREM 2. *If A admits a Type* I *deformation, then some $N_{\alpha\alpha}$ is nonzero.*

Proof. Let $u \in N_K$ satisfy $u * u=u$. We have $u=\sum u_{\gamma\delta}$ with $\gamma, \delta=1, \ldots, s$ where $u_{\gamma\delta}=e_\gamma u e_\delta=e_\gamma * u * e_\delta$. Let $u_{\gamma\delta}$ be a nonzero term in this sum. Repeated multiplications of u by itself allow us to write $u_{\gamma\delta}$ as a sum of terms of the form $u_{\gamma\varepsilon} * u_{\varepsilon\zeta} * \cdots * u_{\omega\delta}$. After a finite number of such multiplications, each such term must involve a repeated subscript; that is, each term has a factor of the form $u_{\alpha\beta} * \cdots * u_{\theta\alpha}$ and not all of these are zero. Thus some $(N_{\alpha\alpha})_K$ contains a nonzero element, and the theorem follows immediately.

Thus the algebra of Example (II) in the previous section could not have admitted a Type I deformation.

We have good reason to believe that the condition of Theorem 2 is not sufficient for Type I deformability, but the evidence would take us too far afield now. See [2]. The search for sufficient conditions partakes, of course, of the general question of existence of deformations, which is far from solution. Thus, it is known that if the Hochschild group $H^3(A, A)=(0)$, then every 2-cocycle of A may be "integrated" to give a (possibly trivial) deformation. And Gerstenhaber has shown that the cup product of commuting derivations of A may be exponentiated in characteristic zero to a deformation. (See [3, p. 64] and [4].) Neither of these results, however, gives explicit information about the production of idempotents. It is possible to concoct various *ad hoc* statements. The following lemma is an example. The proof is straightforward.

LEMMA 3. *Let $z \in N_{\alpha\alpha} \cap$ Ann (N) and $z \notin N^2$. Then there is a deformation of A for which $t^{-1}z$ is an idempotent.*

4. **Deformations of Type II.** Suppose A admits a Type II deformation π_t. What must A look like? What of π_t? Is it equivalent to a deformation of particularly simple form? And what conditions on A guarantee Type II deformability? Theorem 4 below provides an answer to the first question, describing the "size and shape" of the radical N as an S-bimodule. In Theorem 5 we see that the "pre-matrix" part of π_t can essentially be given by polynomials in t; such theorems (see also [1]) assure us that the explicit deformations we construct by trial and

error, adjusting various powers of t, are, in the sense of equivalence, all. And in Theorem 6 we give sufficient conditions for Type II deformability, reducing the problem to that of finding a nilpotent deformation within the radical. For low dimensions this is often practicable. In this regard, the principle enunciated in (ii) below may be useful.

(i) As above k is perfect, $A=N\oplus S$ with unit e, $S=S_1\oplus\cdots\oplus S_s$ is a direct sum of k-matrix algebras, so that $e=e_1+\cdots+e_s$, and $N=\bigoplus N_{\alpha\beta}$ where $N_{\alpha\beta}=e_\alpha Ne_\beta$. We note now that $N_{\alpha\beta}$ is (nonuniquely) the module direct sum of simple S_α, S_β-bimodules $N_{\alpha\beta i}$ (so-called *$\alpha\beta$-blocks*) of dimension $r_\alpha r_\beta$ (recall $r_\alpha=$ rank of S_α).

Now let $e_{\alpha\theta\lambda}$ with θ, $\lambda=1,\dots,r_\alpha$ be the usual matrix basis for S_α. Then each $\alpha\beta$-block $N_{\alpha\beta i}$ has a basis $z_{\alpha\beta i\mu\nu}$ with $\mu=1,\dots,r_\alpha$ and $\nu=1,\dots,r_\beta$ which satisfies $e_{\alpha\theta\lambda}z_{\alpha\beta i\mu\nu}e_{\beta\rho\tau}=\delta_{\lambda\mu}\delta_{\nu\rho}z_{\alpha\beta i\theta\tau}$ (Kronecker deltas). With such a basis for N we observe that if, say, $z_{\alpha\beta h11}z_{\beta\gamma i11}=\sum_j c_{hij}z_{\alpha\gamma j11}$, then also $z_{\alpha\beta h\theta\lambda}z_{\beta\gamma i\lambda\mu}=\sum_j c_{hij}z_{\alpha\gamma j\theta\mu}$ for all θ, λ, μ. That is, the product $z_{\alpha\beta h11}z_{\beta\gamma i11}$ determines $N_{\alpha\beta h}N_{\beta\gamma i}$.

(ii) These considerations lead us to construct the subalgebra (different unit) $A'=N'\oplus S'$ of A as follows: let S' have k-basis $e_{\alpha11}$ (now written as e_α) with $\alpha=1,\dots,s$ as usual, and let N' have k-basis $z_{\alpha\beta i11}$ (now written as $z_{\alpha\beta i}$) with $\alpha,\beta=1,\dots,s$ and one index i for each $N_{\alpha\beta i}$. We multiply in A' as in A. The following folk theorem will be useful.

Reduction Principle. *The subalgebra A' entirely captures the structure of A. Given the ranks $r_1,\dots,r_s$, A can be recovered from A'. Moreover, a straightened-out deformation π_t of A' can be defined immediately on all of A.*

Thus the number s of simple factors S_α, along with the numbers of the various $\alpha\beta$-blocks $N_{\alpha\beta i}$, provide a better index than does dim A to the amount of computation involved in deforming A.

(iii) Now suppose π_t determines a straightened-out Type II deformation of A. Thus $A_t=\operatorname{alg}(V_K,\pi_t)=M_t\oplus T\oplus S_K$, where the subalgebra $T\oplus S_K$ is the Wedderburn factor W_t of A_t. Suppose further that $W_t=\Sigma_1\oplus\Sigma_2\oplus\cdots\oplus\Sigma_\sigma$ is a direct sum of matrix algebras with coefficients in K; we rule out discussion of division algebras over K. Now there is no loss in supposing that the index set $\{1,\dots,s\}$ partitions into disjoint sets $\langle 1\rangle,\langle 2\rangle,\dots,\langle\sigma\rangle$ with $\sigma<s$ such that the simple algebra Σ_p contains the semisimple $S_{\langle p\rangle}=\bigoplus S_\alpha$, $\alpha\in\langle p\rangle$, $p=1,\dots,\sigma$ and, moreover, that rank $\Sigma_p=r_{\langle p\rangle}=$ the sum of those ranks r_α with $\alpha\in\langle p\rangle$. Here, of course, we are identifying the underlying k-space V of A with a subset of V_K, via the basis of (i) above, say. We note $e_{\langle p\rangle}=$ the unit of $\Sigma_p=$ the sum of those e_α with $\alpha\in\langle p\rangle$. And, further, the diagonal blocks of the matrices in Σ_p are filled by the elements of $(S_{\langle p\rangle})_K$.

(iv) Now let the set $\langle p\rangle$ contain more than one element from $\{1,\dots,s\}$. The matrix algebra Σ_p thereby has off-diagonal blocks which are filled by elements of T; these latter were radical elements for the original multiplication π. Cf. z_{12}, z_{21} in Example (II) of §2. In accord with that example, it is entirely straightforward to show: for each ordered pair of distinct $\alpha, \beta \in \langle p \rangle$, there exists an $\alpha\beta$-block $N_{\alpha\beta 0}$ such that, writing $U_p = \bigoplus N_{\alpha\beta 0}$ with distinct $\alpha, \beta \in \langle p \rangle$, the off-diagonal blocks of Σ_p are filled by $(U_p)_K$. Also U_p is an $S_{\langle p \rangle}$-bimodule and the k-algebra $P_p = U_p \oplus S_{\langle p \rangle}$ deforms via π_t into Σ_p. Thus P_p is a *pre-matrix algebra* in the language of [1], that is, a k-algebra with unit which deforms into the algebra of all K-matrices of a given rank, while maintaining the same unit.

And, finally, we form the k-algebra $U = \bigoplus U_p$, $p = 1, \ldots, \sigma$; this is a direct sum of nilpotent ideals. Then we note that $T = U_K$,—in this case the K-subspace T of Theorem 1 does have a k-basis and, moreover, is a subalgebra in the original multiplication.

(v) Now we shall see what the two-sided action of W_t on the radical M_t implies about A. As in Theorem 1, the underlying K-space of M_t is M_K, where M is a k-subspace of the radical N and, in fact, an ideal of A. It follows that $M = \bigoplus M_{pq}$ with $p, q = 1, \ldots, \sigma$, where $M_{pq} = e_{\langle p \rangle} M e_{\langle q \rangle}$ as usual. Further, each M_{pq} is a direct sum of pq-blocks M_{pqi}. And since $e_{\langle p \rangle}$ is the sum of those e_α with $\alpha \in \langle p \rangle$, we may say that $M_{pqi} = \bigoplus N_{\alpha\beta i}$, where $i \geqq 1$ is fixed and the sum is taken over all $\alpha \in \langle p \rangle$, $\beta \in \langle q \rangle$.

Let us define n_{pq} by the formula: K-dim $M_{pq} = n_{pq} r_{\langle p \rangle} r_{\langle q \rangle}$. Thus n_{pq} is the number of pq-blocks in a decomposition of M_{pq}. Now we do some counting. Since each M_{pqi} is composed of all $N_{\alpha\beta i}$ with $i \geqq 1$ and $\alpha \in \langle p \rangle$, $\beta \in \langle q \rangle$, the number of $N_{\alpha\beta i}$ with $i \geqq 1$ must depend only on the index pq and is, in fact, n_{pq}. These considerations, plus recollection of the blocks $N_{\alpha\beta 0}$ in (iv) above, yield the conclusions about dim $N_{\alpha\beta}$ stated in (c) below. We sum up our observations of the last three paragraphs in the following theorem.

THEOREM 4. *Let $A = N \oplus S$ be a finite-dimensional algebra with unit over the perfect field k, with $S = S_1 \oplus \cdots \oplus S_s$ a direct sum of total k-matrix algebras S_α of rank r_α. Let $A_t = M_t \oplus W_t$ be obtained as a Type* II *deformation of A, with $W_t = \Sigma_1 \oplus \cdots \oplus \Sigma_\sigma$ a direct sum of total K-matrix algebras. Then*

(a) *the radical $N = M \oplus U$ where M is a nilpotent ideal in A and U is a nilpotent subalgebra and an S-bimodule; also*

(b) *the index set $\{1, \ldots, s\}$ partitions into sets $\langle 1 \rangle, \ldots, \langle \sigma \rangle$ with $\sigma < s$ such that the subalgebra $U \oplus S$ of A is a direct sum of pre-matrix algebras P_p with $p = 1, \ldots, \sigma$, and*

(c) *there exist integers n_{pq} with $p, q = 1, \ldots, \sigma$ such that*

$$\begin{aligned} \dim_k N_{\alpha\beta} &= r_\alpha r_\beta (1 + n_{pp}), && \textit{for all distinct } \alpha, \beta \in \langle p \rangle, \\ &= (r_\alpha)^2 n_{pp}, && \textit{for } \alpha = \beta \in \langle p \rangle, \\ &= r_\alpha r_\beta n_{pq}, && \textit{for } \alpha \in \langle p \rangle, \beta \in \langle q \rangle \textit{ and } p, q \textit{ distinct.} \end{aligned}$$

(vi) Now we may construct semirigid algebras at will. Form a semisimple $S = S_1 \oplus \cdots \oplus S_s$ and then an S-bimodule N satisfying $N_{\alpha\alpha} = (0)$ and failing one of the dimensional conditions just obtained. Define $N^2 = (0)$. By Theorems 2 and 4, $A = N \oplus S$ is semirigid.

(vii) We continue to suppose that π_t gives a Type II deformation of A, as in Theorem 4. In this paragraph we construct an equivalent deformation of a very simple form,—at least where the multiplication involves elements of W_t. The simplification is accomplished in two steps: the first treats multiplication in W_t, the second the regular action of W_t on the radical M_t.

First, since P_p is a pre-matrix algebra, we know from [1] that there is a k-basis for P'_p (cf. the Reduction Principle of (ii) above) of the form $z_{\alpha\beta 0}$ with $\alpha, \beta \in \langle p \rangle$ (here $z_{\alpha\alpha 0} = e_\alpha$) wherein π_t is equivalent to the multiplication $\Pi_t(z_{\alpha\beta 0}, z_{\beta\gamma 0}) = t^m z_{\alpha\gamma 0}$; here the $m = m_{\alpha\beta\gamma}$ are given as the solution in positive integers of the system

$$(4.1) \qquad m_{\alpha\beta\gamma} + m_{\alpha\gamma\delta} = m_{\alpha\beta\delta} + m_{\beta\gamma\delta} \quad \text{with } m_{\alpha\beta\gamma} = 0 \text{ iff } z_{\alpha\beta 0} z_{\beta\gamma 0} \neq 0.$$

Here $\alpha, \beta, \gamma \in \langle p \rangle$ and $p = 1, \ldots, \sigma$.

It follows, therefore, that there is a K-linear automorphism Φ_t: $V_K \to V_K$ of the form $\Phi_t(x) = x + t\phi_1(x) + \cdots$ which is the identity on M_K such that the composition $\Phi_t \circ \pi_t \circ (\Phi_t^{-1} \times \Phi_t^{-1})$ gives a deformation of π equivalent to π_t and equals the multiplication Π_t in each P_p. We agree to call this new multiplication π_t also.

Now we deal with the left regular action of W_t on M_t determined by π_t. It suffices to consider the action of a matrix algebra Σ_p on a pq-block $(M_t)_{pqi}$. By the Reduction Principle and Theorem 4, therefore, we need only consider the multiplications $\pi_t(z_{\alpha\beta 0}, z_{\beta\gamma i})$ with $\alpha, \beta \in \langle p \rangle$, $\gamma \in \langle q \rangle$ and $i \geqq 1$. Just as in [1], the key observation is that such a product is a scalar (in K) multiple of a single basis element; that is, $\pi_t(z_{\alpha\beta 0}, z_{\beta\gamma i}) = \xi z_{\alpha\gamma i}$ with $\xi = \xi_{\alpha\beta\gamma 0 i}$ an element of $k[[t]]$. We factor $\xi = t^m \kappa$ where $m = m_{\alpha\beta\gamma 0 i}$ and $\kappa = \kappa_{\alpha\beta\gamma 0 i}$ is a power series in t with nonzero constant term. Moreover, it is straightforward that the $z_{\beta\gamma i}$ may be chosen so that $m_{\alpha\beta\gamma 0 i} = 0$ iff $\kappa_{\alpha\beta\gamma 0 i}$ has constant term 1 iff $z_{\alpha\beta 0} z_{\beta\gamma i} = z_{\alpha\gamma i}$ in A'.

Now let us write $\Pi_t(z_{\alpha\beta 0}, z_{\beta\gamma i}) = t^m z_{\alpha\gamma i}$ with $m = m_{\alpha\beta\gamma 0 i}$ as before; it is easy to check that this too defines a left action of the matrix algebra Σ_p on $(M_t)_{pqi}$. Note that we have for $\alpha, \beta \in \langle p \rangle$, $\gamma \in \langle q \rangle$,

$$(4.2) \qquad m_{\alpha\beta\gamma 00} + m_{\alpha\gamma\delta 0 i} = m_{\alpha\beta\delta 0 i} + m_{\beta\gamma\delta 0 i} \quad \text{with } m_{\alpha\beta\gamma 0 i} = 0 \text{ iff } z_{\alpha\beta 0} z_{\beta\gamma i} \neq 0.$$

Here $m_{\alpha\beta\gamma 00} = m_{\alpha\beta\gamma}$ of equation (4.1). Similarly, we define a right action of Σ_q on $(M_t)_{pqi}$ by $\Pi_t(z_{\alpha\beta i}, z_{\beta\gamma 0}) = t^m z_{\alpha\gamma i}$. Now $m = m_{\alpha\beta\gamma i 0}$ and we have, for $\alpha, \beta \in \langle p \rangle$ and $\gamma \in \langle q \rangle$,

$$(4.3) \qquad m_{\alpha\beta\gamma i 0} + m_{\alpha\gamma\delta i 0} = m_{\alpha\beta\delta i 0} + m_{\beta\gamma\delta 00} \quad \text{with } m_{\alpha\beta\gamma i 0} = 0 \text{ iff } z_{\alpha\beta i} z_{\beta\gamma 0} \neq 0.$$

The left and the right actions defined by Π_t commute, so that M_t is a two-sided W_t-module via Π_t, whence the equations

$$(4.4) \qquad m_{\alpha\beta\gamma 0 i} + m_{\alpha\gamma\delta i 0} = m_{\alpha\beta\delta 0 i} + m_{\beta\gamma\delta i 0}.$$

Now since W_t is semisimple, it follows as in Theorem 1 (cf. part (ii) of the proof) that the two-sided actions given by Π_t and π_t are each rigid. We show that they are equivalent in a technical sense akin to that of [3, p. 65]. The same method as in

[1] works here. Thus, let τ be an indeterminate over K and consider the deformation of the Π_t-action given by $\Theta_\tau(z_{\alpha\beta 0}, z_{\beta\gamma i}) = t^m\kappa(\tau)z_{\alpha\gamma i}$ with m and κ as above. By rigidity of Π_t, there exists a $K((\tau))$-linear automorphism Ψ_τ: $M_{K((\tau))} \to M_{K((\tau))}$ of the form $\Psi_\tau(z) = z + \tau\psi_1(z) + \cdots$ such that $\Pi_t(z_{\alpha\beta 0}, z_{\beta\gamma i}) = \Psi_\tau\Theta_\tau(z_{\alpha\beta 0}, \Psi_\tau^{-1}z_{\beta\gamma i})$. Substituting t for τ shows that the actions given by Π_t and π_t are equivalent. Thus we have simplified π_t as follows.

THEOREM 5. *Let A and A_t be as in Theorem* 4, *and let π_t determine the Type* II *deformation. Then*

(a) *the linear equations* (4.1), . . . , (4.4) *are together solvable in positive integers, and*

(b) *there is a k-basis $z_{\alpha\beta i\lambda\mu}$ for A, with $\alpha, \beta = 1, \ldots, s$ and $\lambda = 1, \ldots, r_\alpha$, $\mu = 1, \ldots, r_\beta$, and a deformation Π_t equivalent to π_t which has the simple form $\Pi_t(z_{\alpha\beta i\lambda\mu}, z_{\beta\gamma j\mu\nu}) = t^m z_{\alpha\gamma h\lambda\nu}$ whenever one of i, j is zero. Here $h = \max(i, j)$ and $m = m_{\alpha\beta\gamma ij}$ is given by the solution of the equations in* (a).

Thus the deformation of those multiplications involving the subalgebra $U \oplus S$ of A may always be expressed in terms of polynomials, rather than power series, in t. Also, if the radical is small in the sense that all $n_{pq} \leqq 1$ (cf. Theorem 4), then we claim that the factorization trick may be used in M, so that Type II deformability implies deformability in terms of polynomials in t; there is a $\Pi'_t = \pi + tF_1 + \cdots + t^nF_n$ for some n.

(vii) We have now amassed enough information about Type II deformability to state a sufficiency theorem. This will reduce the problem of deforming A to that of constructing a deformation of a nilpotent ideal M, subject to certain constraints. To see this, we make a further observation about Π_t above. Denote by μ_t the restriction of Π_t to the ideal M of Theorem 4. Also, write $\Pi_t(x, y) = x * y$ for brevity. Now let $u \in U \oplus S$ and $z', z'' \in M$. Then the associativity of the products $u * z' * z''$, $z' * u * z''$, $z' * z'' * u$ imposes constraints on μ_t, considered as a deformation of M. These constraints could be written as explicit formulae involving factors t^m and the structure constants for μ_t, but we refrain from doing this. We shall say in this case that μ_t *associates* with Π_t (with Π_t here considered as a deformation of the action of $U \oplus S$ on M).

The following definition isolates the structural properties of Theorem 4 and will simplify the statement of the theorem. Let $A = N \oplus S$ and $N_{\alpha\beta}$ be as above. A partition $\langle 1\rangle, \langle 2\rangle, \ldots, \langle\sigma\rangle$ of the index set $\{1, \ldots, s\}$ with $\sigma < s$, is said to determine a *Type* II *structure* on N iff each $N_{\alpha\beta}$ decomposes into $\alpha\beta$-blocks $N_{\alpha\beta i}$, with $i = 0, 1, 2, \ldots$ when α, β are distinct elements of some set $\langle p\rangle$ and $i = 1, 2, \ldots$ otherwise, such that (1) $N_{\alpha\beta 0}N_{\beta\gamma i} \subset N_{\alpha\gamma i}$ and $N_{\gamma\beta i}N_{\beta\alpha 0} \subset N_{\gamma\alpha i}$ for all distinct $\alpha, \beta \in \langle p\rangle$ and all γ, and (2) the dimension of $N_{\alpha\beta}$ satisfies (c) of Theorem 4. Given a Type II structure on N, there is a k-basis $z_{\alpha\alpha 0}$ $(=e_\alpha)$ for S', $z_{\alpha\beta i}$ for N' with i as above for which the equations (4.1), . . . , (4.4) have an interpretation (that is, $m_{\alpha\beta\gamma} = 0$ if $z_{\alpha\beta 0}z_{\beta\gamma 0} \neq 0$, otherwise $m_{\alpha\beta\gamma}$ is a positive integer to be determined).

THEOREM 6. *Let* $A = N \oplus S$ *over* k *as in Theorem* 4. *Suppose*

(a) *some partititon* $\langle 1\rangle, \langle 2\rangle, \ldots, \langle\sigma\rangle$ *of* $\{1, \ldots, s\}$ *with* $\sigma < s$ *determines a Type* II *structure on* N, *and*

(b) *the equations* (4.1), ..., (4.4) *have a solution in positive integers.*

Then define $\Pi_t(z_{\alpha\beta i}, z_{\beta\gamma j}) = t^m z_{\alpha\gamma h}$ *whenever one of* i, j *is zero, where* $h = \max(i, j)$ *and* $m = m_{\alpha\beta\gamma ij}$ *is given by the equations above. Suppose further*

(c) *there exists a nilpotent deformation* μ_t *of* $M = \bigoplus N_{\alpha\beta i}$ *with* $i \geqq 1$ *which associates with* Π_t.

Then A *admits a Type* II *deformation.*

This theorem can be used to answer the question: "which k-algebras A deform into the semisimple algebra $\Sigma_1 \oplus \cdots \oplus \Sigma_\sigma$, where each Σ_p is a total K-matrix algebra?" In this case the general problem of constructing a deformation (which involves solving for an infinite number of 2-cochains $F_1, F_2, \ldots$ in sequence) reduces to finding integer solutions for a single finite overdetermined system of linear equations.

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